Rossmoyne Senior High School

Year 12 Trial WACE Examination, 2014

Question/Answer Booklet

MATHEMATICS: SPECIALIST 3C/3D

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words MARKING KEY

Your name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination.

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	50	33⅓
Section Two: Calculator-assumed	13	13	100	100	66⅔
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2013*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil**, except in diagrams.

Section Two: Calculator-assumed **(100 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(5 marks)**

If $f'(x) = 10x\sqrt{3-x}$ and $f(3) = 0$, then $f(x)$ can be written in the form $(ax+b)(3-x)^{\left(\frac{3}{2}\right)}$.

(a) Determine the values of a and b. (3 marks)

Using CAS,

$f(x) = \int f'(x)dx$ ✓

$= 4(3-x)^{\frac{5}{2}} - 20(3-x)^{\frac{3}{2}} + c,\ f(3) = 0 \Rightarrow c = 0$ ✓

Hence

$f(x) = \left(4(3-x) - 20\right)(3-x)^{\frac{3}{2}}$

$= (-4x-8)(3-x)^{\frac{3}{2}} \Rightarrow a = -4,\ b = -8$ ✓

(b) Determine the equation of the tangent to $f(x)$ at the point (2, -16). (2 marks)

$f'(2) = 20$ ✓

$y + 16 = 20(x-2) \Rightarrow y = 20x - 56$ ✓

Question 9 **(5 marks)**

A triangle has vertices at $A(5, -4, -7)$, $B(7, 2, -9)$ and $C(-3, 0, 5)$.

(a) Determine the exact length of side AB. (1 mark)

$$\overrightarrow{AB} = \begin{bmatrix} 7 \\ 2 \\ -9 \end{bmatrix} - \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} \Rightarrow \left|\overrightarrow{AB}\right| = \sqrt{4+36+4} = 2\sqrt{11}$$

✓

(b) Determine the size of $\angle CAB$, correct to the nearest degree. (2 marks)

$$\overrightarrow{AC} = \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \\ 12 \end{bmatrix} \Rightarrow \angle CAB = 99.27 \approx 99^\circ \text{ using CAS}$$

✓ ✓ (Rounding)

(c) The point P lies on the side AC such that the length AP is three times the length PC. Determine the vector $\overrightarrow{OP}$, where O is the origin. (2 marks)

$$\overrightarrow{OP} = \overrightarrow{OA} + \tfrac{3}{4}\overrightarrow{AC}$$ ✓

$$= \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} -8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

✓

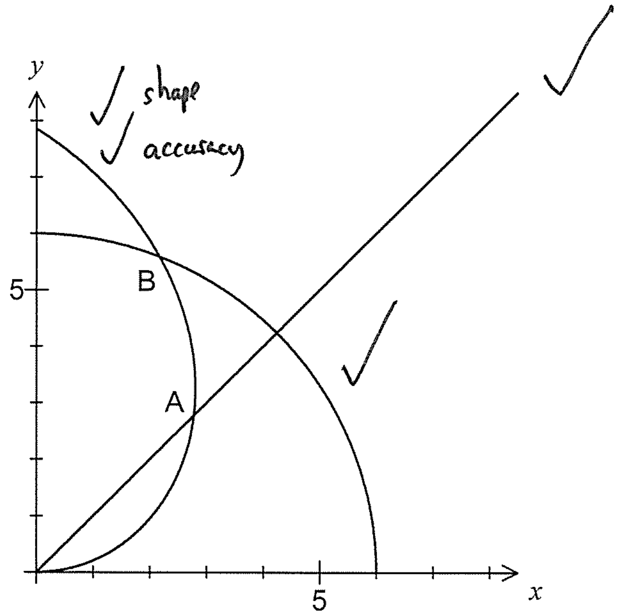
Question 10 **(8 marks)**

(a) Sketch the following polar graphs on the axes below for $0 \le \theta \le \frac{\pi}{2}$.

(i) $\theta = \frac{\pi}{4}$. (1 mark)

(ii) $r = 6$. (1 mark)

(iii) $r = 5\theta$. (2 marks)



(b) Given that $r > 0$ and $0 \le \theta \le \frac{\pi}{2}$, point A is the intersection of $r = 5\theta$ and $\theta = \frac{\pi}{4}$, and point B is the intersection of $r = 5\theta$ and $r = 6$.

Determine the exact polar coordinates of A and B in the form (r, θ). (2 marks)

$A\left(\frac{5\pi}{4}, \frac{\pi}{4}\right)$ ✓

$B\left(6, \frac{6}{5}\right)$ ✓

(c) Determine the distance AB, rounded to three significant figures. (2 marks)

$AB^2 = \left(\frac{5\pi}{4}\right)^2 + 6^2 - 2 \times \left(\frac{5\pi}{4}\right) \times 6 \times \cos\left(\frac{6}{5} - \frac{\pi}{4}\right)$ ✓

$AB = 2.88$ units ✓

Question 11 **(6 marks)**

A triangle has vertices $A(1, 1)$, $B(3, 1)$ and $C(3, 4)$.

(a) Triangle ABC is transformed to $A'(1, -1)$, $B'(3, -1)$ and $C'(3, -4)$. Describe this transformation geometrically and state the 2×2 matrix that will transform ABC to $A'B'C'$. (2 marks)

Reflection in the line $y = 0$. ✓

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ ✓

(b) Triangle $A'B'C'$ is dilated by a scale factor of ten about the origin and then rotated 30° clockwise about the origin to triangle $A''B''C''$.

(i) Determine the single 2×2 matrix that will transform $A'B'C'$ to $A''B''C''$. (2 marks)

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 5\sqrt{3} & 5 \\ -5 & 5\sqrt{3} \end{bmatrix}$$ ✓

✓ order

(ii) Determine the area of triangle $A''B''C''$. (2 marks)

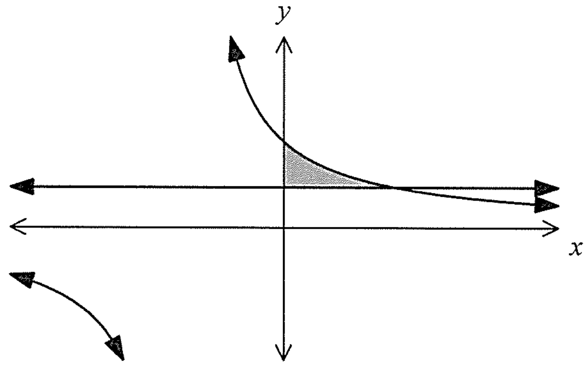
Original area of ABC is 3 sq units.

Reflection and rotation will not change area, but dilation will increase area by a factor of 10^2. ✓

Area is 300 sq units. ✓

Question 12 **(8 marks)**

The line $y = a$ and the curve $xy + 2y = 6$ are shown below. An area in the first quadrant bounded by the line, the curve and the y-axis has been shaded.



(a) The tangents to the curve at $x = -4$ and $x = 4$ intersect at the point (p, q). Determine the values of p and q. (4 marks)

$$y = \frac{6}{x+2} \Rightarrow \frac{dy}{dx} = -\frac{6}{(x+2)^2}$$ ✓

Tangent at $x = 4$: $y = -\frac{x}{6} + \frac{5}{3}$ ✓

Tangent at $x = -4$: $y = -\frac{3x}{2} - 9$ ✓

Solve simultaneously for intersect at (-8, 3) $\Rightarrow p = -8,\ q = 3$ ✓

(b) If the shaded region in the first quadrant, as shown on the diagram above, has an area of 10 square units, determine the value of a, giving your answer to two decimal places. (4 marks)

$$x = \frac{6}{y} - 2 \quad \text{and when } x = 0,\ y = 3$$ ✓

$$\int_a^3 \left(\frac{6}{y} - 2\right) dy = 2a + 6\ln 3 - 6 - \ln(a^6)$$ ✓

$$10 = 2a + 6\ln 3 - 6 - \ln(a^6) \Rightarrow a \approx 0.22$$ ✓

(Ignore $a \approx -0.19$ and $a \approx 12.21$ as these would not give area shown in first quadrant). ✓

Question 13 **(8 marks)**

In a herd of 3 500 cattle, 15 animals are known to have a disease. If left unchecked, the number of diseased cattle in the herd, N, will increase at a rate given by

$$\frac{dN}{dt} = 0.002(3500 - N)$$

where t is the number of days since the initial 15 animals were discovered to have the disease.

(a) Use the above information to write N as a function of t. (3 marks)

$$\int \frac{1}{3500 - N} dN = \int 0.002 dt \quad ✓$$

$$-\ln(3500 - N) = 0.002t + c$$

(or using CAS dSolve function)

$$N = 3500 - ae^{-0.002t}\Big|_{t=0, N=15} \quad ✓$$

$$N = 3500 - 3485e^{-0.002t} \quad ✓$$

(b) How long will it take for more than 4% of the herd to have the disease? (2 marks)

$$3500 \times 0.04 = 3500 - 3485e^{-0.002t}$$

$$t = 18.26 \quad ✓$$

Approximately 18 days. ✓

(c) After 40 days, measures are taken to prevent any more cattle contracting the disease, and the use of medication will decrease the number of diseased cattle so that $\frac{dN}{dt} = -0.15N$. How long will it take from this time for less than 1% of the herd to have the disease? (3 marks)

$$3500 - 3485e^{-0.002 \times 40} = 282.939 \approx 283 \quad ✓$$

$$3500 \times 0.01 = 283e^{-0.15t}$$

$$t = 13.93 \quad ✓$$

Approximately 14 days. ✓

Question 14 **(10 marks)**

The motion of a small body moving in a straight line was recorded by a video camera for 40 seconds. An analysis of the motion showed that the distance, x cm, of the small body from a fixed point O on its path t seconds after recording began was given by $x(t) = 3\cos\frac{\pi t}{4} - 4\sin\frac{\pi t}{4}$.

(a) Show that the body is undergoing simple harmonic motion. (2 marks)

$$x(t) = 3\cos\tfrac{\pi t}{4} - 4\sin\tfrac{\pi t}{4}$$
$$v(t) = x'(t) = \tfrac{\pi}{4}\left(-3\sin\tfrac{\pi t}{4} - 4\cos\tfrac{\pi t}{4}\right)$$
$$a(t) = v'(t) = -\left(\tfrac{\pi}{4}\right)^2\left(3\cos\tfrac{\pi t}{4} - 4\sin\tfrac{\pi t}{4}\right)$$
$$= -\left(\tfrac{\pi}{4}\right)^2 \cdot x(t) \Rightarrow \text{SHM}$$

(b) Determine the initial displacement and velocity of the body. (2 marks)

$$x(0) = 3$$
$$v(0) = -\pi$$

(c) State the period and amplitude of the motion. (2 marks)

$$\text{Period} = \frac{2\pi}{\pi/4} = 8 \text{ seconds}$$

$$\text{Amplitude} = \sqrt{3^2 + 4^2} = 5 \text{ cm}$$

(d) Determine that the maximum speed of the body during its motion. (2 marks)

$$v^2 = \left(\tfrac{\pi}{4}\right)^2\left(5^2 - 0^2\right) \Rightarrow \text{maximum speed is } \tfrac{5\pi}{4} \text{ cm/s}$$

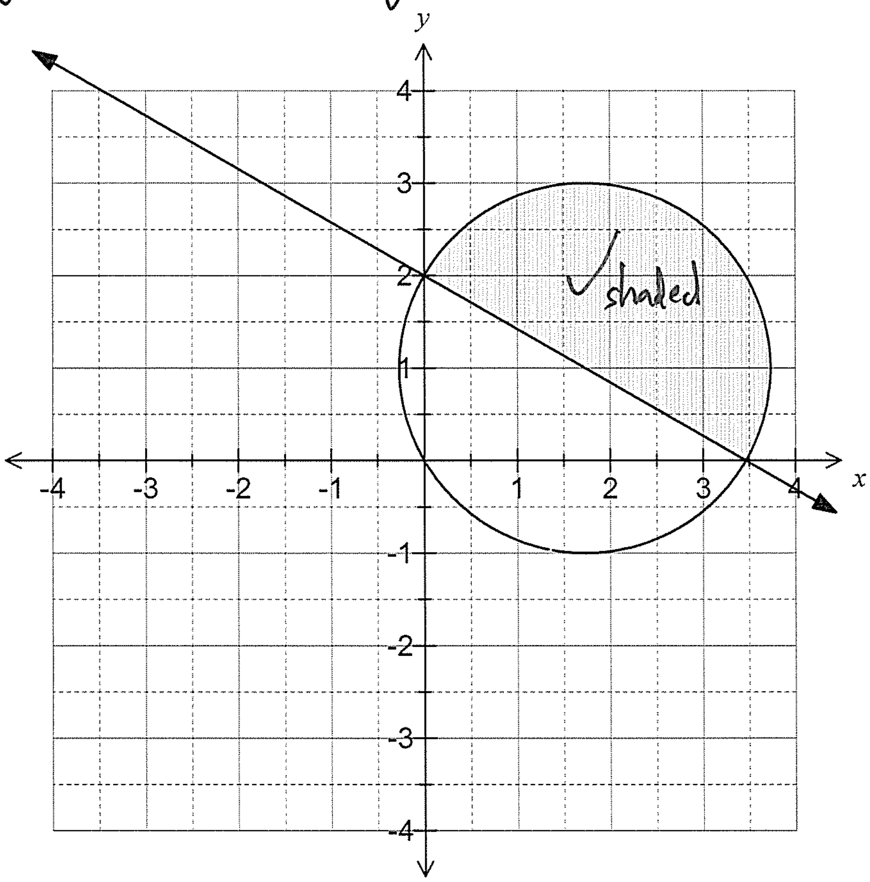
(e) Determine the total distance travelled by the body during the 40 seconds of filming. (2 marks)

In 40 seconds, body will travel 5 complete cycles (period is 8 seconds).
Distance travelled in one cycle is 20 cm, so total distance travelled is 100 cm.

Question 15 **(9 marks)**

(a) Sketch in the complex plane the region satisfying the two inequalities given by

$\left| z - \sqrt{3} - i \right| \leq 2$ and $\left| z \right| \geq \left| z - \sqrt{3} - 3i \right|$ (5 marks)



(b) If z is the complex number that satisfies both inequalities given in (a), determine the minimum and maximum vales of $|z|$. (4 marks)

Minimum value when $z = \frac{1}{2}(\sqrt{3} + 3i) \Rightarrow |z| = \sqrt{3}$.

Maximum value when $z = 2(\sqrt{3} + i) \Rightarrow |z| = 4$.

Question 16 **(8 marks)**

A plane has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda\mathbf{a} + \mu\mathbf{b}$, where $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

(a) If $\mathbf{c} = -2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

(i) show that $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. (1 mark)

$$\begin{bmatrix}-2\\3\\5\end{bmatrix} \bullet \begin{bmatrix}1\\-1\\1\end{bmatrix} = -2-3+5=0 \quad \begin{bmatrix}-2\\3\\5\end{bmatrix} \bullet \begin{bmatrix}2\\3\\-1\end{bmatrix} = -4+9-5=0$$

Hence both perpendicular as dot products are zero. ✓

(ii) determine the equation of the plane in the form $\mathbf{r} \bullet \mathbf{n} = k$. (2 marks)

$$\mathbf{r} \bullet \begin{bmatrix}-2\\3\\5\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \bullet \begin{bmatrix}-2\\3\\5\end{bmatrix} \Rightarrow \mathbf{r} \bullet \begin{bmatrix}-2\\3\\5\end{bmatrix} = 6$$ ✓ ✓

(b) Determine the coordinates of the point where the line $\dfrac{x-1}{2} = \dfrac{y+3}{1} = \dfrac{z-1}{-1}$ meets the plane. (5 marks)

$$\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-1}{-1} = t \Rightarrow x = 1+2t,\ y = -3+t,\ z = 1-t$$ ✓

$$\mathbf{r} = \begin{bmatrix}1\\-3\\1\end{bmatrix} + t\begin{bmatrix}2\\1\\-1\end{bmatrix}$$ ✓

$$\begin{bmatrix}2t+1\\t-3\\1-t\end{bmatrix} \bullet \begin{bmatrix}-2\\3\\5\end{bmatrix} = 6 \Rightarrow -4t-2+3t-9+5-5t=6 \Rightarrow -6t=12 \Rightarrow t=-2$$ ✓ ✓

$$\begin{bmatrix}1\\-3\\1\end{bmatrix} - 2\begin{bmatrix}2\\1\\-1\end{bmatrix} = \begin{bmatrix}-3\\-5\\3\end{bmatrix} \Rightarrow \text{at } (-3, -5, 3)$$ ✓

Question 17 **(8 marks)**

Consider the life cycle of a head louse that has a life span of 4 weeks. In a simplified model, the first week of life is spent as an egg. The egg hatches into a nymph during the second week and for the last two weeks of its life it is an adult, when it can lay several eggs per day. In a model to examine a lice infestation, the following information was assumed.

Age in weeks	0-1 (Egg)	1-2 (Nymph)	2-3 (Adult)	3-4 (Adult)
Eggs laid per week	0	0	30	40
Percentage surviving	20%	20%	10%	0

(a) Write down a Leslie matrix to model changes in the age class distribution from the above information. (2 marks)

$$L = \begin{bmatrix} 0 & 0 & 30 & 40 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

✓ ✓ −1 error/omission

An infestation of lice begins with just two adults, one aged 2-3 weeks and one aged 3-4 weeks in the head of a child.

(b) Determine the expected number of eggs, nymphs and adults in the child's head after

(i) 2 weeks. (2 marks)

✓ $L^2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \\ 0 \\ 0 \end{bmatrix}$ 4 eggs, 14 nymphs and 0 adults. ✓

(ii) 12 weeks. (2 marks)

✓ $L^{12} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.22 \\ 11.06 \\ 4.84 \\ 0.01 \end{bmatrix}$ 9 eggs, 11 nymphs and 5 adults. ✓

(c) If the child commenced a treatment that removed half of the eggs present each week, describe the effect this has on the long term population of head lice. (2 marks)

With the initial model, the population of head lice increases unrealistically after a large number of weeks, ✓ but removing half of the eggs each week leads to the population ceasing to exist after about 30 weeks. ✓

Question 18 **(9 marks)**

Let the A be the area of the region between two concentric circles of radii x and y ($x \geq y$) at any time t seconds. x is increasing at a constant rate of 4 cms^{-1} and when $t = 0$, $x = 60$ cm and $y = 20$ cm.

(a) If y is increasing at a constant rate of 5 cms^{-1}, determine

(i) the rate of increase of A when $t = 0$. (3 marks)

$$A = \pi(x^2 - y^2) \quad ✓$$
$$\frac{dA}{dt} = \pi\left(2x\frac{dx}{dt} - 2y\frac{dy}{dt}\right) \quad ✓$$
$$= \pi(2 \times 60 \times 4 - 2 \times 20 \times 5)$$
$$= 280\pi \text{ cms}^{-1} \quad ✓$$

No penalty for wrong units

(ii) the ratio of x to y when A begins to decrease. (2 marks)

$$\frac{dA}{dt} = 0 \Rightarrow \pi\left(2x\frac{dx}{dt} - 2y\frac{dy}{dt}\right) = 0 \quad ✓$$
$$4x = 5y \Rightarrow x : y = 5 : 4 \quad ✓$$

(iii) the time at which A is zero. (1 mark)

$$x = y$$
$$60 + 4t = 20 + 5t \Rightarrow t = 40 \text{ seconds} \quad ✓$$

(b) If the area A is fixed, determine the rate of increase of y when $x = 90$ cm. (3 marks)

$$\frac{dA}{dt} = 0 \Rightarrow \pi\left(2x\frac{dx}{dt} - 2y\frac{dy}{dt}\right) = 0 \quad ✓$$
$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} \quad ✓$$
$$A_0 = \pi(60^2 - 20^2) = \pi(90^2 - y^2) \Rightarrow y = 70$$
$$\frac{dy}{dt} = \frac{90}{70} \times 4 = \frac{36}{7} \text{ cms}^{-1} \quad (\approx 5.143) \quad ✓$$

No need for units

Question 19 **(7 marks)**

(a) If $u = \frac{x}{t}$, determine $\frac{du}{dt}$ and the values of u when $t = x$ and when $t = \frac{x}{y}$. (2 marks)

$\frac{du}{dt} = -\frac{x}{t^2}$ ✓

$t = x,\ u = 1$ and $t = \frac{x}{y},\ u = y$ ✓ both

(b) Explain why $\int_1^{x/y} \frac{1}{t}\,dt = \int_1^{x} \frac{1}{t}\,dt + \int_x^{x/y} \frac{1}{t}\,dt$ for all $x, y > 0$. (2 marks)

From the graph of $y = \frac{1}{t}$ in the first quadrant, it is clear that the area under the curve between 1 and x added to the area between x and $\frac{x}{y}$ will be the same as the area between 1 and $\frac{x}{y}$. ✓ ✓

be lenient here → any reasonable statement

(c) The natural logarithm of x can be expressed as $\ln x = \int_1^x \frac{1}{t}\,dt$ for $x > 0$. Use this definition, together with the substitution $u = \frac{x}{t}$, to prove that $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$ for all $x, y > 0$. Do **not** use any laws of logarithms. (3 marks)

$$LHS = \ln\left(\frac{x}{y}\right)$$

$$= \int_1^{x/y} \frac{1}{t}\,dt$$

$$= \int_1^{x} \frac{1}{t}\,dt + \int_x^{x/y} \frac{1}{t}\,dt$$

From (a), $\frac{1}{t} = \frac{u}{x}$ and $dt = -\frac{x}{u^2}du \Rightarrow \frac{1}{t}dt = -\frac{1}{u}$ ✓

$$= \left[\ln|t|\right]_1^x - \int_1^y \frac{1}{u}du$$ ✓

$$= \ln(x) - \ln(1) - \left[\ln|u|\right]_1^y$$

$$= \ln(x) - (\ln(y) - \ln(1))$$ ✓

$$= \ln(x) - \ln(y)$$

$$= RHS$$

Question 20 **(9 marks)**

(a) Prove by contradiction that the last digit of 2^n, where n is a positive integer, will never be zero. (3 marks)

Assume that the last digit of 2^n is 0, so that $2^n = 10k$, where k is an integer.

Since $10k$ is divisible by 5, then so must 2^n, but this is impossible, and so our original assumption is contradicted, meaning that the last digit of 2^n can never be zero.

(b) The sequence of hexagonal numbers, H_n, is given by the recursive rule $H_{n+1} = H_n + 4n + 1$, $H_1 = 1$.

(i) Show that the third hexagonal number is 15. (1 mark)

$$\begin{aligned} T_2 &= T_1 + 4(1) + 1 \\ &= 6 \\ T_3 &= T_2 + 4(2) + 1 \\ &= 15 \end{aligned}$$

(ii) Prove by induction that the n^{th} hexagonal number, H_n, can also be found using the explicit rule $H_n = 2n^2 - n$, $n \ge 1$. (5 marks)

When $n = 1$, $H_1 = 2(1)^2 - 1 = 1$, and so the rule is true for this case.

If it is assumed that the rule is true for $n = k$, i.e. $H_k = 2k^2 - k$, then it is necessary to prove the result for $n = k + 1$, i.e. $H_{k+1} = 2(k+1)^2 - (k+1)$.

Now, using the original recursive rule gives:

$$\begin{aligned} H_{k+1} &= H_k + 4k + 1 \\ &= 2k^2 - k + 4k + 1 \\ &= 2k^2 + 4k + 2 - k - 1 \\ &= 2(k+1)^2 - (k+1) \end{aligned}$$

Thus shown to be true for $n = 1$ and as the truth for $n = k$ implies the result for $n = k + 1$ it follows that the conjecture is true for all positive integers.

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

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